Fibonacci numbers, Euler's 2-periodic continued fractions and moment sequences *

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Abstract

We prove that certain sequences of finite continued fractions associated with a 2-periodic continued fraction with period $a, b > 0$ are moment sequences of discrete signed measures supported in the interval $[-1, 1]$, and we give necessary and sufficient conditions in order that these measures are positive. For $a = b = 1$ this proves that the sequence of ratios F_{n+1}/F_{n+2}, $n \geq 0$ of consecutive Fibonacci numbers is a moment sequence.

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1 Introduction

The last chapter of Euler's masterpiece *Introductio in Analysin Infinitorum* (vol. I), in English version [6], is devoted to continued fractions. Euler considered there 1- and 2-periodic continued fractions to get rational approximations to

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square roots of natural numbers. We say that a continued fraction of the form

$$\cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \ddots}}}} \tag{1.1}$$

is k-periodic if $a_{j+k} = a_j$ for $j = 1, 2, \ldots$.

If a natural number n is sum of two squares $n = m^2 + l^2$ of natural numbers, then for $a = 2m/l$ the convergents of the 1-periodic continued fraction

$$\cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \ddots}}}} \tag{1.2}$$

give rational approximations for $\sqrt{n}$. Indeed, this continued fraction converges to the positive root of $x^2 + ax - 1$:

$$x = \frac{-a + \sqrt{a^2+4}}{2} = -\frac{m}{l} + \frac{\sqrt{n}}{l}.$$

For example, for $n = 5$, Euler took $m = 1, l = 2$, and so $a = 1$; the continued fraction (1.2) converges then to $(\sqrt{5} - 1)/2$. In this example, the rational approximations are

$$\frac{0}{1}, \quad \frac{1}{1}, \quad \frac{1}{2}, \quad \frac{2}{3}, \quad \frac{3}{5}, \quad \frac{5}{8}, \quad \cdots \tag{1.3}$$

which are ratios of consecutive Fibonacci numbers $F_n, n \geq 0$: $F_0 = 0, F_1 = 1, F_{n+1} = F_n + F_{n-1}, n \geq 1$. For a relation between quotients of consecutive Fibonacci numbers and electrical networks see [11, p.43].

In [9] Ismail introduced a generalization $F_n(\theta)$ of the Fibonacci numbers as the solution to the difference equation

$$x_{n+1} = 2\sinh\theta\, x_n + x_{n-1}, \quad n \geq 1, \tag{1.4}$$

with the initial conditions $x_0 = 0, x_1 = 1$. Here θ can be any positive real number. The generalized Fibonacci numbers are natural numbers, when $2\sinh\theta$ is a natural number, and the Fibonacci numbers correspond to $2\sinh\theta = 1$.

When n is not a sum of two squares, Euler considered a 2-periodic continued fraction of the form

$$\cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{a + \cfrac{1}{b + \ddots}}}} \tag{1.5}$$

to get rational approximations for $\sqrt{n}$. To do that, Euler was implicitly using the Pell equation $m^2n = d^2 - 1$. That equation was a proposal of Fermat to the British mathematicians. It was studied by Wallis and Brouncker. Euler thought that Brouncker's results, published by Wallis in his *Algebra*, were due to Pell, and baptized it *Pell's equation*. Euler also studied Pell's equation, but it was Lagrange who solved it completely. Assuming that n is not a square of a natural number, the Pell equation $m^2n = d^2 - 1$ has always infinitely many solutions $m, d \in \mathbb{N}$. By taking positive integers a, b for which $ab = 2d - 2$, the 2-periodic continued fraction above gives rational approximations for $\sqrt{n}$. Indeed, that continued fraction converges to the positive root of $ax^2 + abx - b$:

$$x = \frac{-ab + \sqrt{a^2b^2 + 4ab}}{2a} = -\frac{d-1}{a} + \frac{m\sqrt{n}}{a}.$$

For instance, for $n = 7$, Euler took $m = 3, d = 8, a = 2, b = 7$, and the continued fraction (1.5) converges then to $(-7 + 3\sqrt{7})/2$.

For a 2-periodic continued fraction (1.5), where a, b are positive real numbers, we associate the sequence $(s_n(a, b, w))_n$ of finite continued fractions (to simplify the notation we remove the dependence on a, b and w):

$$s_0 = w, \quad s_1 = \frac{1}{a+w}, \quad s_2 = \frac{1}{a + \cfrac{1}{b+w}}, \quad s_3 = \frac{1}{a + \cfrac{1}{b + \cfrac{1}{a+w}}}, \quad \cdots \tag{1.6}$$

where $w \geq 0$. For $w = 0$ we obtain the sequence of convergents to (1.5). Since

$$s_{n+2} = \frac{1}{a + \cfrac{1}{b + s_n}},$$

we find

$$s_n = \frac{N_n}{D_n}, \quad n \geq 0,$$

where the sequences $(N_n)_n$ and $(D_n)_n$ can be defined recursively:

$$N_{n+2} = bD_n + N_n, \quad n \geq 0, \tag{1.7}$$

$$D_{n+2} = abD_n + aN_n + D_n, \quad n \geq 0, \tag{1.8}$$

with initial conditions $N_0 = w$, $N_1 = 1$, $D_0 = 1$, $D_1 = a + w$.

When $a = b$, by setting $D_{-1} = w$, we actually have $N_n = D_{n-1}$, $n \geq 0$, and then the sequence $(D_n)_n$ can be defined recursively in the form

$$D_{n+1} = aD_n + D_{n-1}, \quad n \geq 0, \tag{1.9}$$

with initial conditions $D_{-1} = w$, $D_0 = 1$. This is the same difference equation as (1.4). The general difference equation of second order with constant coefficients $x_{n+1} = ax_n + bx_{n-1}$ has been studied by Kalman and Mena in [10].

The moments μ_n, $n \geq 0$, of a positive Borel measure μ on the real line are defined by $\mu_n = \int_{\mathbb{R}} t^n d\mu(t)$, $n \geq 0$, assuming that these integrals are finite. Sequences that are moments of positive measures on $[0,\infty)$ were characterized by Stieltjes in his fundamental memoir [13] by certain quadratic forms being non-negative. In [7] Hamburger extended the results of Stieltjes to moment sequences of positive measures concentrated on the whole real line. Later Hausdorff, cf. [8], characterized moment sequences of measures concentrated on the unit interval $[0,1]$ by complete monotonicity, see also [1], [14], [4]. Moment sequences of positive measures on the interval $[-1,1]$ can be characterized as bounded Hamburger moment sequences, i.e. bounded sequences $(s_n)_n$ such that all the Hankel matrices

$$\mathcal{H}_n = (s_{i+j})_{i,j=0}^n\,, \quad n = 0, 1, \ldots \tag{1.10}$$

are positive semidefinite.

The purpose of this note is to characterize the sequences (1.6) of finite continued fractions $(s_n)_n$, which are bounded moment sequences.

In the following δ_c denotes the Dirac measure having the mass 1 at the point $c \in \mathbb{R}$.

Theorem 1.1. *For $a, b, w \in \mathbb{R}$, $a, b > 0, w \geq 0$, define*

$$q = \frac{2 + ab - \sqrt{a^2b^2 + 4ab}}{2}, \quad \alpha = \frac{q(aw - (1-q))}{qaw + 1 - q}, \quad \beta = \frac{qa - (1-q)w}{a + (1-q)w}. \tag{1.11}$$

Then $0 < q < 1, |\alpha|, |\beta| < 1$ and $(s_n(a,b,w))_n$ is the sequence of moments of the discrete signed measure ρ supported in $[-1,1]$ and defined by

$$\rho = \frac{1}{a}(1-q)\delta_1 + \frac{1}{2a}\left(\frac{1}{q} - q\right)\sum_{k=0}^{\infty}\left(\alpha^{k+1} + \beta^{k+1}\right)\delta_{q^{k+1}} + \left(\alpha^{k+1} - \beta^{k+1}\right)\delta_{-q^{k+1}}. \tag{1.12}$$

The measure ρ is positive if and only if $a \geq b$,

$$w \geq -b/2 + \sqrt{(b/2)^2 + b/a}, \quad \textit{and} \quad w \geq -(a+b)/4 + \sqrt{((a+b)/4)^2 + 1}.$$

In particular for $a \geq b$ and $w = 1$, the measure ρ is always positive. Taking $a = b = 2\sinh\theta > 0$, and $w = 1/a$ we get the following result:

Corollary 1.2. *The sequences*

$$\frac{F_{n+1}(\theta)}{F_{n+2}(\theta)}, \quad \frac{F_{n+3}(\theta)}{F_{n+2}(\theta)}, \; n \geq 0 \tag{1.13}$$

of quotients of generalized Fibonacci numbers $F_n(\theta)$ defined by (1.4) *are moment sequences of the measures μ_θ and $\nu_\theta = (2\sinh\theta)\delta_1 + \mu_\theta$, where*

$$\mu_\theta = e^{-\theta}\delta_1 + 2\cosh\theta \sum_{k=1}^{\infty} e^{-4k\theta}\delta_{(-1)^k e^{-2k\theta}}. \tag{1.14}$$

In particular for $2\sinh\theta = 1$ the sequence $F_{n+1}/F_{n+2}, n \geq 0$ is the moments of the probability measure

$$\mu = \varphi\delta_1 + \sqrt{5}\sum_{k=1}^{\infty} \varphi^{4k}\delta_{(-1)^k\varphi^{2k}}, \tag{1.15}$$

where $\varphi = (\sqrt{5}-1)/2$.

Remark 1.3. Using the so-called Binet formula for the Fibonacci numbers, it is easy to see that $F_{n+1}, n \geq 0$ is a moment sequence of the measure

$$\tau = \frac{\sqrt{5}+1}{2\sqrt{5}}\delta_{(1+\sqrt{5})/2} + \frac{\sqrt{5}-1}{2\sqrt{5}}\delta_{(1-\sqrt{5})/2}.$$

A similar formula holds for the generalized Fibonacci numbers of Ismail, see [9, formula (2.2)].

Remark 1.4. It was proved in [3] that $F_\alpha/F_{n+\alpha}$, $n \geq 0$ is the moment sequence of a signed measure μ_α with total mass 1. Here α is a natural number and the signed measure μ_α is a probability measure precisely when α is an even number. The orthogonal polynomials corresponding to μ_α are little q-Jacobi polynomials, where $q = (1-\sqrt{5})/(1+\sqrt{5})$. The results were used to prove Richardson's formula for the elements in the inverse of the Filbert matrix $(1/F_{1+i+j})$, cf. [12]. These results were extended to generalized Fibonacci numbers in [9]. For an extension to quantum integers see [2].

2 Proofs

Proof of Theorem 1.1. First of all, we find a closed expression for the denominators $(D_n)_n$ of the finite continued fractions $(s_n)_n$ defined in (1.6).

From (1.7) and (1.8) we get

$$N_n = \frac{D_n - D_{n-2}}{a}, \quad n \geq 2. \tag{2.1}$$

This recurrence can also be extended to $n = 0, 1$ by defining $D_{-2} = 1 - aw$ and $D_{-1} = w$. Inserting (2.1) in (1.8), we find that

$$D_{n+2} = (2 + ab)D_n - D_{n-2}, \quad n \geq 0, \tag{2.2}$$

with initial conditions $D_{-2} = 1 - aw$, $D_{-1} = w$, $D_0 = 1$ and $D_1 = a + w$.

That means that the sequences $(D_{2n})_n$ and $(D_{2n+1})_n$ are both solutions of the difference equation

$$x_{n+1} = (2 + ab)x_n - x_{n-1}, \quad n \geq 0,$$

with initial conditions $x_{-1} = 1 - aw, w$, $x_0 = 1, a + w$, respectively. Any solution of this difference equation has the form $c_0 q_0^n + c_1 q_1^n$, where q_0 and q_1 are the solutions of $x^2 - (2 + ab)x + 1 = 0$. We write

$$q = \frac{2 + ab - \sqrt{a^2b^2 + 4ab}}{2}, \tag{2.3}$$

so that q and $1/q$ are the solutions of $x^2 - (2 + ab)x + 1 = 0$, and $0 < q < 1$. We then have that there exist numbers c_0, c_1, d_0, d_1 such that

$$D_{2n} = c_0 q^{-n} + c_1 q^n, \quad n \geq -1, \tag{2.4}$$

$$D_{2n+1} = d_0 q^{-n} + d_1 q^n, \quad n \geq -1. \tag{2.5}$$

Using the initial conditions $D_{-2} = 1 - aw$, $D_{-1} = w$, $D_0 = 1$ and $D_1 = a + w$, it is easy to calculate that

$$c_0 = \frac{1 - q + qaw}{1 - q^2}, \qquad c_1 = \frac{q(1 - q - aw)}{1 - q^2}, \tag{2.6}$$

$$d_0 = \frac{a + (1 - q)w}{1 - q^2}, \qquad d_1 = \frac{q((1 - q)w - qa)}{1 - q^2}. \tag{2.7}$$

Note that $c_0, d_0 > 0$. Writing $\alpha = -c_1/c_0$ and $\beta = -d_1/(qd_0)$, we find

$$\alpha = \frac{q(aw - (1 - q))}{qaw + 1 - q}, \quad \beta = \frac{qa - (1 - q)w}{a + (1 - q)w}, \tag{2.8}$$

and it is clear that $|\alpha|, |\beta| < 1$ because $a > 0, 0 < q < 1$ and $w \geq 0$.

We need to establish some technical properties of α and β, which we collect in

Lemma 2.1. **1.** *$\alpha \geq 0$ if and only if $w \geq -b/2 + \sqrt{(b/2)^2 + b/a}$.*

2. *Assume $w > 0$. Then $-\beta \leq \alpha$ if and only if $a \geq b$.*

3. *If $w = 0$ then $\beta = -\alpha = q$.*

4. *$\alpha \geq \beta$ if and only if $w \geq -(a + b)/4 + \sqrt{((a + b)/4)^2 + 1}$.*

Proof. 1. By (2.8) we have that $\alpha \geq 0$ is equivalent to $q + aw - 1 \geq 0$, hence to

$$1 - aw \leq q = \frac{2 + ab - \sqrt{a^2b^2 + 4ab}}{2},$$

or

$$\sqrt{a^2b^2 + 4ab} \leq 2aw + ab,$$

which is equivalent to $aw^2 + abw - b \geq 0$ because $a > 0$. Since $a, b > 0$ and $w \geq 0$, we finally get that $\alpha \geq 0$ if and only if

$$w \geq -b/2 + \sqrt{(b/2)^2 + b/a}.$$

2. According to (2.8), $-\beta \leq \alpha$ if and only if

$$\frac{(1-q)w - aq}{a + (1-q)w} \leq \frac{q(aw - (1-q))}{qaw + 1 - q},$$

and a straightforward computation gives that this is equivalent to

$$w(1+q)(q^2 - (2+a^2)q + 1) \leq 0.$$

Using that q satisfies $q^2 - (2+ab)q + 1 = 0$, the left-hand side of this inequality can be reduced to $w(1+q)aq(b-a)$, and for $w > 0$ the result follows.

3. Follows by inspection.

4. We similarly get that $\alpha \geq \beta$ if and only if

$$w^2 + w(a+b)/2 - 1 \geq 0,$$

which is equivalent to the given condition because $w \geq 0$. $\square$

We now continue the proof of Theorem 1.1.

Using (2.1), we can write

$$s_n = \frac{N_n}{D_n} = \frac{1}{a}\frac{D_n - D_{n-2}}{D_n} = \frac{1}{a}\left(1 - \frac{D_{n-2}}{D_n}\right).$$

From the formulas (2.4) and (2.6), and taking into account that $\alpha = -c_1/c_0$, we have

$$\begin{aligned}
\frac{D_{2n-2}}{D_{2n}} &= \frac{c_0 q^{-n+1} + c_1 q^{n-1}}{c_0 q^{-n} + c_1 q^n} = q\frac{1 - \alpha q^{2n-2}}{1 - \alpha q^{2n}} \\
&= q(1 - \alpha q^{2n-2}) \sum_{k=0}^{\infty} \alpha^k q^{2nk} \\
&= q\left(1 + (1 - q^{-2}) \sum_{k=0}^{\infty} \alpha^{k+1} q^{2n(k+1)}\right),
\end{aligned}$$

showing that $s_{2n} = \int t^{2n} d\mu$, $n \geq 0$, where the measure μ is defined as

$$\mu = \frac{1}{a}(1-q)\delta_1 + \frac{1}{a}\left(\frac{1}{q} - q\right)\sum_{k=0}^{\infty} \alpha^{k+1}\delta_{q^{k+1}}. \tag{2.9}$$

In a similar way, it can be proved that $s_{2n+1} = \int t^{2n+1} d\nu$, $n \geq 0$, where the measure ν is defined as

$$\nu = \frac{1}{a}(1-q)\delta_1 + \frac{1}{a}\left(\frac{1}{q} - q\right)\sum_{k=0}^{\infty} \beta^{k+1}\delta_{q^{k+1}}. \tag{2.10}$$

Take now the reflected measures $\check{\mu}$ and $\check{\nu}$ of μ and ν with respect to the origin:

$$\begin{aligned}
\check{\mu} &= \frac{1}{a}(1-q)\delta_{-1} + \frac{1}{a}\left(\frac{1}{q} - q\right)\sum_{k=0}^{\infty} \alpha^{k+1}\delta_{-q^{k+1}}, \\
\check{\nu} &= \frac{1}{a}(1-q)\delta_{-1} + \frac{1}{a}\left(\frac{1}{q} - q\right)\sum_{k=0}^{\infty} \beta^{k+1}\delta_{-q^{k+1}}.
\end{aligned}$$

The measure $\rho = (\mu + \check{\mu})/2 + (\nu - \check{\nu})/2$ has then the same even moments as μ and the same odd moments as ν. That is, the n'th moment of ρ is just s_n, $n \geq 0$. A simple computation shows that the measure ρ is given by (1.12).

It is clear that the measure ρ is positive if and only if

$$\alpha^{k+1} + \beta^{k+1},\ \alpha^{k+1} - \beta^{k+1} \geq 0, \quad k \geq 0,$$

i.e. if and only if $\alpha \geq 0$ and $-\alpha \leq \beta \leq \alpha$. The second part of Theorem 1.1 follows now by applying Lemma 2.1.

□

Proof of Corollary 1.2. For $a = b = 2\sinh\theta > 0$ and $w = 1/a$ we see that the three conditions of Theorem 1.1 are satisfied, so $s_n = s_n(a, a, 1/a)$ is a moment sequence of a positive measure $\rho = \mu_\theta$. We prove that $s_n = F_{n+1}(\theta)/F_{n+2}(\theta)$ by induction. This formula holds for $n = 0$ by inspection and clearly $s_{n+1} = 1/(a + s_n)$. We therefore find, assuming the formula for a fixed n

$$s_{n+1} = \frac{1}{a + F_{n+1}(\theta)/F_{n+2}(\theta)} = \frac{F_{n+2}(\theta)}{aF_{n+2}(\theta) + F_{n+1}(\theta)} = \frac{F_{n+2}(\theta)}{F_{n+3}(\theta)}.$$

A small calculation shows that $\alpha = -\beta = q^2, q = e^{-2\theta}$, $e^{-\theta} = (1-q)/a$ and $\mu_\theta = \rho$ defined in (1.12) is given by (1.14). The formula for ν_θ follows easily from the recurrence equation for $F_n(\theta)$. □

3 Concluding remarks

Euler did not use 3-periodic continued fractions (nor any other periodicity bigger than 2) to find rational approximations of square roots of natural numbers: to use 3-periodic continued fractions is not more useful than to use 1-periodic continued ones, and to use 4-periodic continued fractions is not more useful than to use 2-periodic ones, and so forth, cf. [6, p.322].

One can consider finite continued fractions like (1.6) for a 3-periodic continued fraction of positive periods a, b, c. Using the same approach as before, we can find three signed measures μ_0, μ_1 and μ_2 on $[-1, 1]$ such that the $(3n+i)$'th moment of μ_i is equal to s_{3n+i}, $i = 0, 1, 2$, $n \geq 0$. However, we are not able to construct a measure on $\mathbb{R}$ from these three measures having its $(3n+i)$'th moment equal to the n'th moment of μ_i, $i = 0, 1, 2$. The same happens for k-periodicity when $k > 2$.

By a result of Boas, cf. [14, p. 138], see also [5], any sequence is a moment sequence of a signed measure. On the other hand we know that the Hankel determinants of the moments of a positive measure are non-negative. Computations indicate that if we consider the sequence $(s_n(a, a, c, 1))_n$, then some of the Hankel determinants are negative except when $c = a$.

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